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Analysis of Spatial Curved Bi-Fixed Beam with Varying Curvature and Varying Cross-Sectional Area Using Finite Displacement Transfer Method

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ABSTRACT

This article is focused on determination of fixed-end reactions, displacements, internal forces and internal moments for the spatial curved bi-fixed beam with varying curvature and varying cross-sectional area. The objective is to analyze spatially curved beam without involving analytical differentiation and integration of the governing equations. A finite displacement transfer method to analyze spatially curved beam is used to obtain numerically approximate analysis results and to eliminate analytical differentiation and integration, completely. The results of the analysis procedure are compared with the results of other methods found in the literature. For the specific case of the circular helical beam, the fixed end reactions' values have absolute maximum and minimum percentage differences of 1.23% and 0.06%, respectively. For the specific case of the elliptic-helical beam, the fixed end reactions' values have absolute maximum and minimum percentage differences of 1.13% and 0.0%, respectively. The finite difference method, along with the recursive scheme, gives reasonably accurate results without involving complex calculations.

Key words : Finite difference, Fixed end reaction, Internal force, Internal moment.

1. INTRODUCTION

Curved beams are used in various civil engineering structures such as stair, balcony, etc. Most of the civil engineering structural elements are statically indeterminate. Basically there are two methods of the structural analysis namely classical methods and matrix methods. A detailed classification of the analysis methods is presented by Sushanta Ghuku and Kashi Nath Saha [1]. Matrix methods are very popular due to easiness in computer application. Analysis of the spatial curved beam is complex due to combined actions of axial force, shear forces, bending moments, twisting moment and different geometry of the curved beam axis. Fixed end reactions are the prime requirement for the analysis of the curved beam using matrix methods. The Bernoulli–Euler and Timoshenko theories can be employed for the analysis of the curved beam [1], [2]. The equilibrium equations, compatibility equations and constitutive relationships or energy principles can be used to represent mechanical behavior of the spatial curved beam [3]-[5]. In any case, analytical solution requires integration of the governing equations to determine displacements and fixed end reactions. Hence, analytical solutions can be obtained for some restricted shapes of the curved beam axis, such as circular and parabolic shapes [6]-[8]. In most of the cases, numerical methods must be employed for the solution of the governing equations of the curved beam [9]-[12]. Researchers use different approaches such as Transfer Matrix Method [9]-[11], [13], [14], Finite Difference Method [15], [16], Finite Element Method [17]-[21] for the analysis of either planer or spatial curved beam to obtain displacements, reactions and internal forces. Finite element method is one of the approximate methods of the analysis. Accuracy of the result depends on the number of finite elements, but size of the problem also increases.

It is easy to determine displacements and reactions of the circular and parabolic curved beam using analytical exact method because analytical exact integration of the governing equations of such types of curved beam are possible. Elliptical curved beam is a classic example for which exact displacements and reactions can be not determined using analytical exact method [4], [22]. It is essential to employ numerical method when the curved beam axis is arbitrary or with different geometrical equations. Analysis of the planer or spatial curved beam includes calculation of differentiation and integration.

Objective of the present paper is to determine fixed end reactions, displacements, internal forces and moments for the spatial curved bi-fixed beam with varying curvature and varying cross-sectional area along the curved beam axis.

This paper presents an automated calculation procedure to analyze spatial curved bi-fixed beam. The finite displacement transfer method used to analyze spatial curved beam eliminates analytical differentiation and integration completely. Also, there is no need to develop problem-specific differential equations for the spatial curved beam. The main disadvantage of the proposed calculation method is to supply coordinates of the nodes of the curved beam axis. Shear deformations are included in the present study, but local buckling phenomenon is not in the scope of the present study.

Following assumptions are made in the finite displacement transfer method to analyze spatial curved fixed beam. (i) The material is elastic and homogeneous. (ii) The cross-section of the curved beam has two axes of symmetry so that twisting moment and bending moment occurs independent of one another. (iii) Every cross-section remains undistorted during deformation. (iv) The cross-sectional dimensions of the curved beam are small in comparison to the length of the curved beam. (v) Average values of the cross-sectional properties of discretized element are considered.

2. LITERATURE REVIEW

E. Tufekci and O. Y. Dogruer [6] presented analytical expressions to determine displacements of the planer arches with varying curvature and cross-section. The analytical exact solution is possible only when the arch axis represented by same geometrical equation throughout its length and analytical exact integration of the differential equation possible.

Gimena *et al.* [7] derived stiffness matrix and equivalent load vector of a 3D curved beam with variable cross-section using Transfer Matrix Method. The numerical solution procedure presented using fourth order Runge-Kutta method. Gimena *et al.* [8] presented a system of twelve ordinary differential equations to analyze curved beam considering different shape geometry of the curved centroidal line, shearing deformation, varying cross-sectional area, non-symmetric section and generalized loads. Internal forces and displacements of the circular arch, circular balcony and circular helical beam were obtained based on exact analytical method. Gimena *et al.* [9], [10] presented twelve ordinary differential equations for non-naturally curved beam. Exact analytical and forth order Runge-Kutta numerical procedure proposed to obtain stiffness matrix and transfer matrix. Stiffness matrix and equivalent load vector were obtained using transfer matrix. Gimena *et al.* [11] presented first order, second order and forth order Runge-Kutta numerical method for the analysis of arbitrary curved beam.

Analysis methods proposed in the literature [7]-[10] needed equations of the tangent, normal, binormal, bending curvature and flexion curvature of the curved beam. Hence, it may be very difficult to analyze the spatial curved beam with different geometry in different segments of the curved beam axis.

Wankui and Hui [12] expressed the governing equations in terms of displacement functions and used finite difference scheme to obtain numerical solution for the planer curved beam.

Arici and Granata [13] obtained displacements and internal forces of space curved bar with generalized Winkler soil using Transfer Matrix Method. They had presented excellent recurrence scheme to relate initial node and final node of the curved beam. This recurrence scheme used to determine the unknowns based on the boundary conditions, but it requires rearrangement of the total matrix.

Sarria *et al.* [14] presented a unique system of equations associated to the curved beam with elastic supports by joining the twelve equations of the stiffness matrix expression with the twelve equations of support conditions. A linear system of ordinary differential equations solved using Finite Transfer Method.

Al-Azzawi and Shaker [15] extended Timoshenko's deep beam theory to derive governing differential equations of curved deep beams resting on elastic foundations with both compressional and frictional resistances considering linear elastic behavior. Finite Difference Equations were introduced in the governing differential equations to obtain deflections and bending moment.

Jirásek *et al.* [16] presented formulation of first-order differential equations of a geometrically nonlinear beam element using Bernoulli beam element that can accommodate arbitrarily large rotations of cross sections. These equations were discretized by finite differences and shooting method used to convert the boundary value problem into an initial value problem.

Previous researchers have used various numerical methods to obtain solution of the governing differential equations. Although some researchers have presented rigorous mathematical formulas for the analysis of curved beams such as circular, parabolic, and elliptical, this is very difficult for computer applications of generalized spatially curved beams.

3. BASIC FORMULATIONS

3.1 Geometry of Spatial Curved Beam Axis

Figure 1 shows geometry of the spatial curved beam axis with reference to the Cartesian coordinate system xyz, which is also global axis of the spatial curved beam. Local axis (Member axis) is represented by *tnb* i.e. tangent, normal and binormal directions. The spatial curved beam axis may be arbitrary or with single or multiple geometrical equations along the length of the curve.

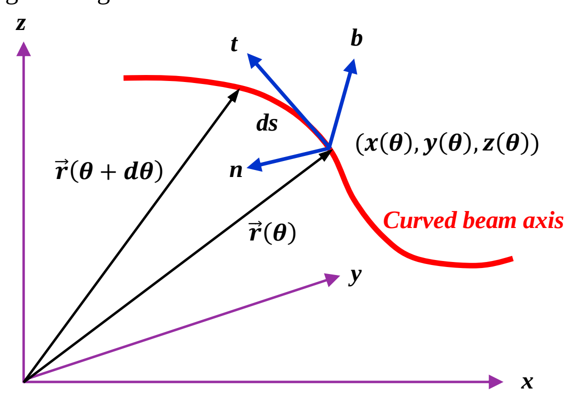

Figure 1: Geometry of the spatial curved beam axis.

Cartesian coordinate of any point on spatial curved beam axis is represented by $(x(\theta), y(\theta), z(\theta))$. Here, θ may be any independent parameter.

The $\vec{r}(\theta)$ and $\vec{r}(\theta + d\theta)$ are the position vectors corresponding to θ and $(\theta + d\theta)$ respectively. The interval $d\theta$ may be positive or negative.

Position vector can be expressed as follows:

$$\vec{r}(\theta) = x(\theta)I + y(\theta)J + z(\theta)K \tag{1}$$

Where, I, J and K are the unit vectors in the global directions x, y and z respectively.

Direction of the movement can be represented as,

$$\frac{dr}{d\theta} = \frac{dx}{d\theta}I + \frac{dy}{d\theta}J + \frac{dz}{d\theta}K \tag{2}$$

If total length and elemental arc length of the curved beam axis are denoted by s and ds respectively, then

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \tag{3}$$

Where, dx, dy and dz are the projected length of the elemental arc in the global x, y and z directions respectively.

Speed of the movement can be represented as,

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} \tag{4}$$

Unit tangent,

$$T = T_1\, I + T_2\, J + T_3\, K \tag{5}$$

T_1, T_2 and T_3 are resolved parts of unit tangent T in global x, y and z directions respectively and can be expressed as,

$$T_1 = \frac{dx}{d\theta}\Big/\frac{ds}{d\theta} \tag{6}$$

$$T_2 = \frac{dy}{d\theta}\Big/\frac{ds}{d\theta} \tag{7}$$

$$T_3 = \frac{dz}{d\theta}\Big/\frac{ds}{d\theta} \tag{8}$$

Unit Normal,

$$N = N_1\, I + N_2\, J + N_3\, K \tag{9}$$

N_1, N_2 and N_3 are resolved parts of unit normal N in global x, y and z directions respectively and can be expressed as,

$$N_1 = \frac{\left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2x}{d\theta^2} - \frac{dx}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]}{\left\{\begin{array}{l} \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2x}{d\theta^2} - \frac{dx}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \\ + \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \\ + \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \end{array}\right\}^{1/2}} \tag{10}$$

$$N_2 = \frac{\left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]}{\left\{\begin{array}{l} \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2x}{d\theta^2} - \frac{dx}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \\ + \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \\ + \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \end{array}\right\}^{1/2}} \tag{11}$$

$$N_3 = \frac{\left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]}{\left\{\begin{array}{l} \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2x}{d\theta^2} - \frac{dx}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \\ + \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \\ + \left[\left(\frac{ds}{d\theta}\right)^2 \frac{d^2z}{d\theta^2} - \frac{dz}{d\theta}\left[\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\frac{d^2y}{d\theta^2} + \frac{dz}{d\theta}\frac{d^2z}{d\theta^2}\right]\right]^2 \end{array}\right\}^{1/2}} \tag{12}$$

Unit binormal,

$$B = T \times N \tag{13}$$

$$B = B_1\, I + B_2\, J + B_3\, K \tag{14}$$

B_1, B_2 and B_3 are resolved parts of unit binormal B in global x, y and z directions respectively and can be expressed as,

$$B_1 = (T_2N_3 - T_3N_2) \tag{15}$$

$$B_2 = (T_3N_1 - T_1N_3) \tag{16}$$

$$B_3 = (T_1N_2 - T_2N_1) \tag{17}$$

Above equations of tangent, normal and binormal can be used to obtain analytical exact results. Following equations of tangent and normal instead of equations (6), (7), (8), (10), (11) and (12) shall be used to determine approximate results using numerical method.

$$T_1 = \frac{dx}{ds} \tag{18}$$

$$T_2 = \frac{dy}{ds} \tag{19}$$

$$T_3 = \frac{dz}{ds} \tag{20}$$

$$N_1 = \frac{\left[(ds)^2 \cdot d^2x - dx[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]}{\left\{\begin{array}{l} \left[(ds)^2 \cdot d^2x - dx[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \\ + \left[(ds)^2 \cdot d^2y - dy[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \\ + \left[(ds)^2 \cdot d^2z - dz[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \end{array}\right\}^{1/2}} \tag{21}$$

$$N_2 = \frac{\left[(ds)^2 \cdot d^2y - dy[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]}{\left\{\begin{array}{l} \left[(ds)^2 \cdot d^2x - dx[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \\ + \left[(ds)^2 \cdot d^2y - dy[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \\ + \left[(ds)^2 \cdot d^2z - dz[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \end{array}\right\}^{1/2}} \tag{22}$$

$$N_3 = \frac{\left[(ds)^2 \cdot d^2z - dz[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]}{\left\{\begin{array}{l} \left[(ds)^2 \cdot d^2x - dx[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \\ + \left[(ds)^2 \cdot d^2y - dy[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \\ + \left[(ds)^2 \cdot d^2z - dz[dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z]\right]^2 \end{array}\right\}^{1/2}} \tag{23}$$

The tangent, normal and binormal directions are continuously moving along axis of curved beam. The rotation of the local axis with reference to global axis can be represented by the rotation of axis matrix,

$$[R] = \begin{bmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \tag{24}$$

3.2 Finite Difference Scheme

To implement finite difference numerical method, the spatial curved beam axis is discretized into n number of segments as shown in figure 2.

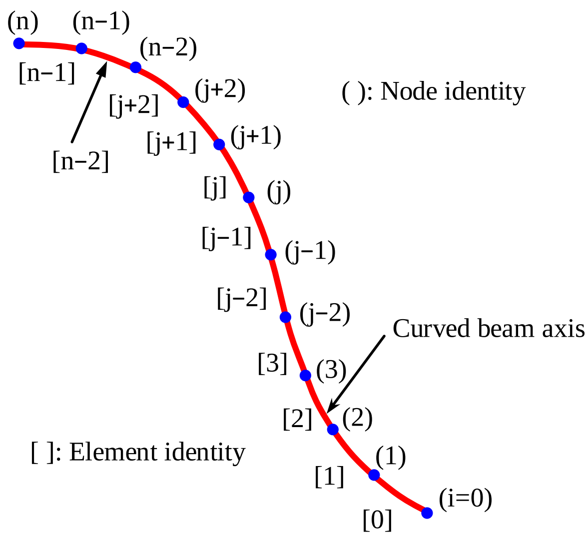

Figure 2: Discretization of the Spatial Curved Beam Axis

In this article, Central Difference formula adopted to obtain approximate solution.
Consider any node i with coordinate *(x(θ), y(θ), z(θ))* on the curved beam axis and interval d*θ*.

$$x(\theta) = x_i \tag{25}$$
$$y(\theta) = y_i \tag{26}$$
$$z(\theta) = z_i \tag{27}$$
$$dx = \frac{(x_{i+1} - x_{i-1})}{2} \tag{28}$$
$$dy = \frac{(y_{i+1} - y_{i-1})}{2} \tag{29}$$
$$dz = \frac{(z_{i+1} - z_{i-1})}{2} \tag{30}$$

4. METHODS AND FORMULATIONS

Matrix formulation of fixed end reactions, displacements, internal forces and moments for the spatial curved beam with both ends fixed is presented in this section. Finite Displacement Transfer method is used for the formulations.
Figure 3 shows discretized spatial curved bi-fixed beam axis, external forces, external moments, fixed ends reactions.
Following are the properties of the curved beam: *E* is the Young's modulus of elasticity; *G* is the shear modulus; *A* is the cross-sectional area; I_t is the torsion constant; I_n and I_b are the moment of inertia; ξ_n and ξ_b are the shear coefficients.

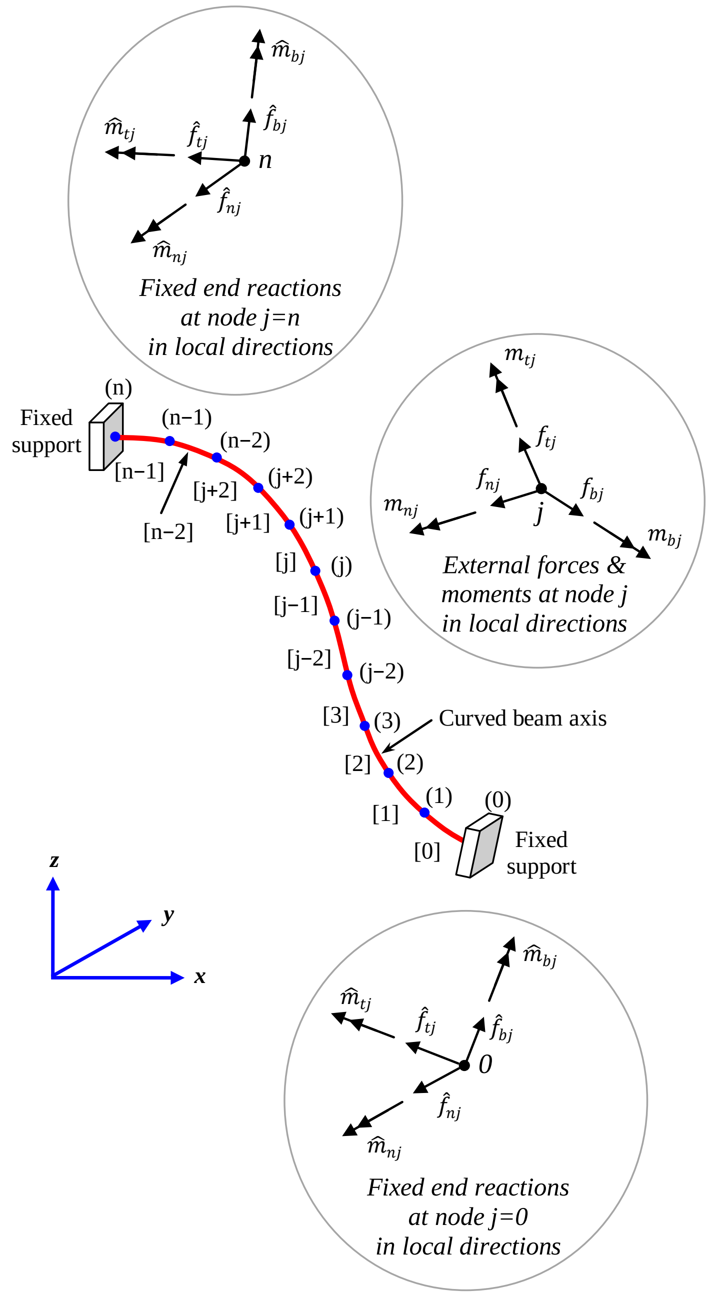

Figure 3: Spatial Curved Bi-Fixed Beam

If curved beam subjected to varying load, then load intensity at the node *i* is considered to obtain energy equivalent nodal load at the node *i*. In case of varying cross-sectional properties such as area, moment of inertia, torsion constant, shear coefficients of the element, the average cross-sectional properties are considered.
Matrices of external forces and moments at node *j* (*j = 0, 1, 2, 3, …, n*) in the member directions (local axis) are denoted by $[f_{mj}]$ and $[m_{mj}]$, respectively.

$$[f_{mj}] = [f_{tj} \quad f_{nj} \quad f_{bj}]^T \tag{31}$$
$$[m_{mj}] = [m_{tj} \quad m_{nj} \quad m_{bj}]^T \tag{32}$$

f_{tj}, f_{nj} and f_{bj} are the external loads at node *j* in tangent, normal and binormal directions, respectively.
m_{tj}, m_{nj} and m_{bj} are the external moments at node *j* about

tangent, normal and binormal directions, respectively.

If external loads are in global directions, the external loads should be transformed into member directions using the following relationship:

$$\begin{bmatrix} f_{mj} \\ m_{mj} \end{bmatrix} = \begin{bmatrix} R_j & O \\ O & R_j \end{bmatrix} \begin{bmatrix} f_{sj} \\ m_{sj} \end{bmatrix} \tag{33}$$

f_{sj} and m_{sj} are the matrices of the external loads at node *j* in global directions (structure axis).

$$[f_{sj}] = [f_{xj} \quad f_{yj} \quad f_{zj}]^T \tag{34}$$

$$[m_{sj}] = [m_{xj} \quad m_{yj} \quad m_{zj}]^T \tag{35}$$

f_{xj}, f_{yj} and f_{zj} are the external forces at node *j* in *x*, *y* and *z* directions, respectively.

m_{xj}, m_{yj} and m_{zj} are the external moments at node *j* about *x*, *y* and *z* directions, respectively.

Matrices of fixed ends reactions at node *j* (*j = 0, n*) in the member directions (local axis) are denoted by $[\hat{f}_{mj}]$ and $[\widehat{m}_{mj}]$, respectively.

$$[\hat{f}_{mj}] = [\hat{f}_{tj} \quad \hat{f}_{nj} \quad \hat{f}_{bj}]^T \tag{36}$$

$$[\widehat{m}_{mj}] = [\widehat{m}_{tj} \quad \widehat{m}_{nj} \quad \widehat{m}_{bj}]^T \tag{37}$$

$\hat{f}_{tj}$, $\hat{f}_{nj}$ and $\hat{f}_{bj}$ are the fixed end reactions at node *j* in tangent, normal and binormal directions, respectively.

$\widehat{m}_{tj}$, $\widehat{m}_{nj}$ and $\widehat{m}_{bj}$ are the fixed end reactions at node *j* in tangent, normal and binormal directions, respectively.

Matrices of internal forces and moments at node *j* (*j = 0, 1, 2, 3, …, n*) in the member directions (local axis) are denoted by $[F_{mj}]$ and $[M_{mj}]$, respectively.

$$[F_{mj}] = [F_{tj} \quad F_{nj} \quad F_{bj}]^T \tag{38}$$

$$[M_{mj}] = [M_{tj} \quad M_{nj} \quad M_{bj}]^T \tag{39}$$

F_{tj} is the axial force at node *j* in tangent direction.

F_{nj} and F_{bj} are the shear forces at node *j* in normal and binormal directions, respectively.

M_{tj} is the twisting moment (torsional moment) at node *j* in tangent direction.

M_{nj} and M_{bj} are the bending moments at node *j* in normal and binormal directions, respectively.

The spatial curved bi-fixed beam is statically indeterminate of sixth degree. Therefore, principle of superposition and compatibility conditions are used to analyze the spatial curved bi-fixed beam. In the formulations, two imaginary nodes are considered, one just before the initial node *0* and another just after the last node *n*.

4.1 Displacements of Statically Determinate Curved Beam Subjected to External Loads

First, consider the statically determinate spatial curved beam subjected to external loads. Node *0* is freely supported and node *n* is fixed.

Internal forces and moments at node *j* are obtained using recursive formula [13] as given below:

$$\begin{bmatrix} F_{mj} \\ M_{mj} \end{bmatrix} = \begin{bmatrix} f_{mj} \\ m_{mj} \end{bmatrix} + \begin{bmatrix} R_j & O \\ O & R_j \end{bmatrix} \begin{bmatrix} I & O \\ C_{j,(j-1)} & I \end{bmatrix} \begin{bmatrix} R_{(j-1)}^T & O \\ O & R_{(j-1)}^T \end{bmatrix} \begin{bmatrix} F_{m(j-1)} \\ M_{m(j-1)} \end{bmatrix} \tag{40}$$

R_j is the rotation matrix of node *j* and can be obtained as explained in Section II. $R_{(j-1)}^T$ is the transpose of rotation matrix of node *j−1*. I is the unit matrix of size 3 × 3. O is the null matrix of size 3 × 3. Coordinate transformation matrix $C_{j,(j-1)}$ is given below:

$$[C_{j,(j-1)}] = \begin{bmatrix} 0 & -(z_{(j-1)} - z_j) & (y_{(j-1)} - y_j) \\ (z_{(j-1)} - z_j) & 0 & -(x_{(j-1)} - x_j) \\ -(y_{(j-1)} - y_j) & (x_{(j-1)} - x_j) & 0 \end{bmatrix} \tag{41}$$

Now, consider *j*th element on the curved beam axis. Node *j* and node *j+1* are the two nodes of the *j*th element. Finite displacements at node *j*, relative to those at node *j+1*, due to external forces and moments at node *j* can be obtained using following equation [23].

$$\begin{bmatrix} d\delta_{mj} \\ d\varphi_{mj} \end{bmatrix} = \begin{bmatrix} D_F & O \\ O & D_M \end{bmatrix} \begin{bmatrix} F_{mj} \\ M_{mj} \end{bmatrix} ds \tag{42}$$

Finite linear displacements at node *j* in member directions (local axis),

$$[d\delta_{mj}] = [d\delta_{tj} \quad d\delta_{nj} \quad d\delta_{bj}]^T \tag{43}$$

$d\delta_{tj}$, $d\delta_{nj}$ and $d\delta_{bj}$ are the finite linear displacements at node *j* in tangent, normal and binormal directions, respectively.

Finite angular displacements at node *j* about member directions (local axis),

$$[d\varphi_{mj}] = [d\varphi_{tj} \quad d\varphi_{nj} \quad d\varphi_{bj}]^T \tag{44}$$

$d\varphi_{tj}$, $d\varphi_{nj}$ and $d\varphi_{bj}$ are the finite angular displacements at node *j* about tangent, normal and binormal directions, respectively.

D_F and D_M are matrices of the elastic and geometrical properties,

$$[D_F] = \begin{bmatrix} 1/EA & 0 & 0 \\ 0 & \xi_n/GA & 0 \\ 0 & 0 & \xi_b/GA \end{bmatrix} \tag{45}$$

$$[D_M] = \begin{bmatrix} 1/GI_t & 0 & 0 \\ 0 & 1/EI_n & 0 \\ 0 & 0 & 1/EI_b \end{bmatrix} \tag{46}$$

Finite displacements in member directions can be transformed into global directions using following equation.

$$\begin{bmatrix} d\delta_{sj} \\ d\varphi_{sj} \end{bmatrix} = \begin{bmatrix} R_j^T & O \\ O & R_j^T \end{bmatrix} \begin{bmatrix} d\delta_{mj} \\ d\varphi_{mj} \end{bmatrix} \tag{47}$$

R_j^T is the transpose of rotation matrix of node j

Finite linear displacements at node *j* in global directions (structure directions),

$$[d\delta_{sj}] = [d\delta_{xj} \quad d\delta_{yj} \quad d\delta_{zj}]^T \tag{48}$$

$d\delta_{xj}$, $d\delta_{yj}$ and $d\delta_{zj}$ are the finite linear displacements at node *j* in *x*, *y* and *z* directions, respectively.

Finite angular displacements at node *j* about global directions (structure directions),

$$[d\varphi_{sj}] = [d\varphi_{xj} \quad d\varphi_{yj} \quad d\varphi_{zj}]^T \tag{49}$$

$d\varphi_{xj}$, $d\varphi_{yj}$ and $d\varphi_{zj}$ are the finite angular displacements at node *j* about *x*, *y* and *z* directions, respectively.

Kinematic equivalent finite displacements at node *i* may be obtained from finite displacements at *j* by finite displacement transfer relationship [23],

$$\begin{bmatrix} d\delta_{sji} \\ d\varphi_{sji} \end{bmatrix} = \begin{bmatrix} I & C_{ij} \\ O & I \end{bmatrix} \begin{bmatrix} d\delta_{sj} \\ d\varphi_{sj} \end{bmatrix} \tag{50}$$

$d\delta_{sji}$ is the matrix of transformed finite linear displacements at node *i* in global directions,

$$[d\delta_{sji}] = [d\delta_{xji} \quad d\delta_{yji} \quad d\delta_{zji}]^T \tag{51}$$

$d\delta_{xji}$, $d\delta_{yji}$ and $d\delta_{zji}$ are the transformed finite linear displacements at node *i* in *x*, *y* and *z* directions, respectively.
$d\varphi_{sji}$ is the matrix of transformed finite angular displacements at node *i* about global directions,

$$[d\varphi_{sji}] = [d\varphi_{xji} \quad d\varphi_{yji} \quad d\varphi_{zji}]^T \tag{52}$$

$d\varphi_{xji}$, $d\varphi_{yji}$ and $d\varphi_{zji}$ are the transformed finite angular displacements at node *i* about *x*, *y* and *z* directions, respectively.

$$[C_{ij}] = \begin{bmatrix} 0 & (z_i - z_j) & -(y_i - y_j) \\ -(z_i - z_j) & 0 & (x_i - x_j) \\ (y_i - y_j) & -(x_i - x_j) & 0 \end{bmatrix} \tag{53}$$

It is observed that, $[C_{ij}] = -[C_{ji}]$
Cumulative displacements in global directions at node *i* can be obtained using following equation [24].

$$\begin{bmatrix} \delta_{si} \\ \varphi_{si} \end{bmatrix} = \sum_{j=i}^{(n-1)} \begin{bmatrix} d\delta_{sji} \\ d\varphi_{sji} \end{bmatrix} \qquad (i = 0, 1, 2, 3, \dots, n) \tag{54}$$

δ_{si} is the matrix of total linear displacements at node *i* in global directions,

$$[\delta_{si}] = [\delta_{xi} \quad \delta_{yi} \quad \delta_{zi}]^T \tag{55}$$

δ_{xi}, δ_{yi} and δ_{zi} are the total linear displacements at node *i* in *x*, *y* and *z* directions, respectively.
φ_{si} is the matrix of total angular displacements at node *i* about *x*, *y* and *z* directions, respectively.

$$[\varphi_{si}] = [\varphi_{xi} \quad \varphi_{yi} \quad \varphi_{zi}]^T \tag{56}$$

φ_{xi}, φ_{yi} and φ_{zi} are the total angular displacements at node *i* about *x*, *y* and *z* directions, respectively.
Using Eq. (54), total displacements at free end (node *0*) can be obtained by keeping *i=0*.
Total displacements at node *i* in local directions (member directions) can be obtained using rotation matrix as given below:

$$\begin{bmatrix} \delta_{mi} \\ \varphi_{mi} \end{bmatrix} = \begin{bmatrix} R_i & O \\ O & R_i \end{bmatrix} \begin{bmatrix} \delta_{si} \\ \varphi_{si} \end{bmatrix} \tag{57}$$

δ_{mi} is the matrix of total linear displacements at node *i* in local directions,

$$[\delta_{mi}] = [\delta_{ti} \quad \delta_{ni} \quad \delta_{bi}]^T \tag{58}$$

δ_{ti}, δ_{ni} and δ_{bi} are the total linear displacements at node *i* in tangent, normal and binormal directions, respectively.
φ_{mi} is the matrix of total angular displacements at node *i* about local directions.

$$[\varphi_{mi}] = [\varphi_{ti} \quad \varphi_{ni} \quad \varphi_{bi}]^T \tag{59}$$

φ_{ti}, φ_{ni} and φ_{bi} are the total angular displacements at node *i* about tangent, normal and binormal directions, respectively.
$[\Delta_{m0}]_L$ stand for the total displacements at node *0* in local directions (member directions) due to the external loads. Hence,

$$[\Delta_{m0}]_L = \begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_L \tag{60}$$

4.2 Displacements of Statically Determinate Spatial Curved Beam Due to Reactions

Again, consider statically determinate spatial curved beam with node *n* fixed and node *0* freely supported. The curved beam is loaded with reactions at node *0*. The reactions at node *0* are redundant; therefore unit load method is used to formulate equation of displacements due to these redundant.
Matrices of the unit load applied at node 0 corresponding to the redundant are as follows:

$$\begin{bmatrix} \hat{f}_{t0} = 1 \\ \hat{f}_{n0} = 0 \\ \hat{f}_{b0} = 0 \\ \widehat{m}_{t0} = 0 \\ \widehat{m}_{n0} = 0 \\ \widehat{m}_{b0} = 0 \end{bmatrix}, \begin{bmatrix} \hat{f}_{t0} = 0 \\ \hat{f}_{n0} = 1 \\ \hat{f}_{b0} = 0 \\ \widehat{m}_{t0} = 0 \\ \widehat{m}_{n0} = 0 \\ \widehat{m}_{b0} = 0 \end{bmatrix}, \begin{bmatrix} \hat{f}_{t0} = 0 \\ \hat{f}_{n0} = 0 \\ \hat{f}_{b0} = 1 \\ \widehat{m}_{t0} = 0 \\ \widehat{m}_{n0} = 0 \\ \widehat{m}_{b0} = 0 \end{bmatrix},$$

$$\begin{bmatrix} \hat{f}_{t0} = 0 \\ \hat{f}_{n0} = 0 \\ \hat{f}_{b0} = 0 \\ \widehat{m}_{t0} = 1 \\ \widehat{m}_{n0} = 0 \\ \widehat{m}_{b0} = 0 \end{bmatrix}, \begin{bmatrix} \hat{f}_{t0} = 0 \\ \hat{f}_{n0} = 0 \\ \hat{f}_{b0} = 0 \\ \widehat{m}_{t0} = 0 \\ \widehat{m}_{n0} = 1 \\ \widehat{m}_{b0} = 0 \end{bmatrix}, \begin{bmatrix} \hat{f}_{t0} = 0 \\ \hat{f}_{n0} = 0 \\ \hat{f}_{b0} = 0 \\ \widehat{m}_{t0} = 0 \\ \widehat{m}_{n0} = 0 \\ \widehat{m}_{b0} = 1 \end{bmatrix}$$

Internal forces and moments at node *j* can be obtained by modifying Eq. (40) as given below:

$$\begin{bmatrix} F_{mj} \\ M_{mj} \end{bmatrix} = \begin{bmatrix} \hat{f}_{mj} \\ \widehat{m}_{mj} \end{bmatrix} \qquad (j = 0) \tag{61}$$

$$\begin{bmatrix} F_{mj} \\ M_{mj} \end{bmatrix} = \begin{bmatrix} R_j & O \\ O & R_j \end{bmatrix} \begin{bmatrix} I & O \\ C_{j,(j-1)} & I \end{bmatrix} \begin{bmatrix} R_{(j-1)}^T & O \\ O & R_{(j-1)}^T \end{bmatrix} \begin{bmatrix} F_{m(j-1)} \\ M_{m(j-1)} \end{bmatrix} \qquad (j = 1, 2, 3, \dots, n) \tag{62}$$

Displacements at node *0* due to unit loads can be obtained using Eq. (57). The resulting matrix of displacements due to unit loads is given below:

$$[\Delta_{m0}]_{UL} = \left[\begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_{\hat{f}_{t0}=1} \begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_{\hat{f}_{n0}=1} \begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_{\hat{f}_{b0}=1} \begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_{m_{t0}=1} \begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_{m_{n0}=1} \begin{bmatrix} \delta_{m0} \\ \varphi_{m0} \end{bmatrix}_{m_{b0}=1} \right] \tag{63}$$

Reactions at note *0* and node *n* may be written as,

$$[\mathcal{R}_0] = \begin{bmatrix} \hat{f}_{m0} \\ \widehat{m}_{m0} \end{bmatrix} \quad and \quad [\mathcal{R}_n] = \begin{bmatrix} \hat{f}_{mn} \\ \widehat{m}_{mn} \end{bmatrix} \tag{64}$$

Total displacements at node *0* due to reactions at node 0 are,

$$[\Delta_{m0}]_{\mathcal{R}_0} = [\Delta_{m0}]_{UL}[\mathcal{R}_0] \tag{65}$$

4.3 Reactions and Displacements of Spatial Curved Bi-Fixed Beam

Reactions of curved bi-fixed beam can be obtained using compatibility conditions and equilibrium conditions. Node *0* is fixed in the curved bi-fixed beam; therefore, displacements at node *0* are null.

$$[\Delta_{m0}]_L + [\Delta_{m0}]_{\mathcal{R}_0} = [O]$$
$$\therefore [\Delta_{m0}]_L + [\Delta_{m0}]_{UL}[\mathcal{R}_0] = [O]$$
$$\therefore [\mathcal{R}_0] = -[\Delta_{m0}]_{UL}^{-1}[\Delta_{m0}]_L \tag{66}$$

Now, apply following equations to obtain internal forces and moments.

$$\begin{bmatrix} F_{mj} \\ M_{mj} \end{bmatrix} = \begin{bmatrix} \hat{f}_{mj} \\ \hat{m}_{mj} \end{bmatrix} + \begin{bmatrix} f_{mj} \\ m_{mj} \end{bmatrix} \qquad (j = 0) \tag{67}$$

$$\begin{bmatrix} F_{mj} \\ M_{mj} \end{bmatrix} = \begin{bmatrix} f_{mj} \\ m_{mj} \end{bmatrix} + \begin{bmatrix} R_j & O \\ O & R_j \end{bmatrix} \begin{bmatrix} I & O \\ C_{j,(j-1)} & I \end{bmatrix} \begin{bmatrix} R^T_{(j-1)} & O \\ O & R^T_{(j-1)} \end{bmatrix} \begin{bmatrix} F_{m(j-1)} \\ M_{m(j-1)} \end{bmatrix} \qquad (j = 1, 2, 3, \dots, n) \tag{40}$$

At node *n*, negative of the internal forces and moments are the reactions at node *n*.

$$[\mathcal{R}_n] = \begin{bmatrix} \hat{f}_{mn} \\ \hat{m}_{mn} \end{bmatrix} = - \begin{bmatrix} F_{mn} \\ M_{mn} \end{bmatrix} \tag{68}$$

Total displacements at node *i* in member directions can be obtained using Eq. (57); rewritten,

$$\begin{bmatrix} \delta_{mi} \\ \varphi_{mi} \end{bmatrix} = \begin{bmatrix} R_i & O \\ O & R_i \end{bmatrix} \begin{bmatrix} \delta_{si} \\ \varphi_{si} \end{bmatrix} \tag{57}$$

5. RESULTS AND DISCUSSIONS

Two examples, circular helical stair beam with varying cross-sectional area [14] and elliptic–helical beam [9] are considered for the verification of the formulation of finite displacement transfer method for the analysis of spatial curved bi-fixed beam. A C++ computer program is developed to obtain calculation results based on the formulation presented in this article.

5.1 Circular Helical Stair Beam with Varying Cross-Sectional Area

A circular helical bi-fixed beam of the stair is shown in figure 4. Circular helical beam has a radius of 1.0 m, a height of 3.0 m and a total angle rotated of π rad.

The beam cross-section is varying through the curved beam axis; initial c/s diameter d_0 = 0.10 m and final c/s diameter d_n = 0.20 m.

Shearing coefficients are ξ_n = 1.33 and ξ_b = 1.33.

The material of the helical beam is the same through the curved beam axis, with elastic modulus E = 206.01 kN/mm^2 and shear modulus G = 79.23 kN/mm^2.

A uniformly distributed load 8.0 kN/m^2, including its own weight, applied on the 1.0 m stair width throughout the curved beam length. This uniformly distributed load creates two types of actions throughout the curved beam axis, as given below:

Force (kN/m)

$f_x = 0, f_y = 0, f_z = 8$

Moment (kNm/m)

$m_x = -4 \sin\theta, m_y = 4 \cos\theta, m_z = 0$

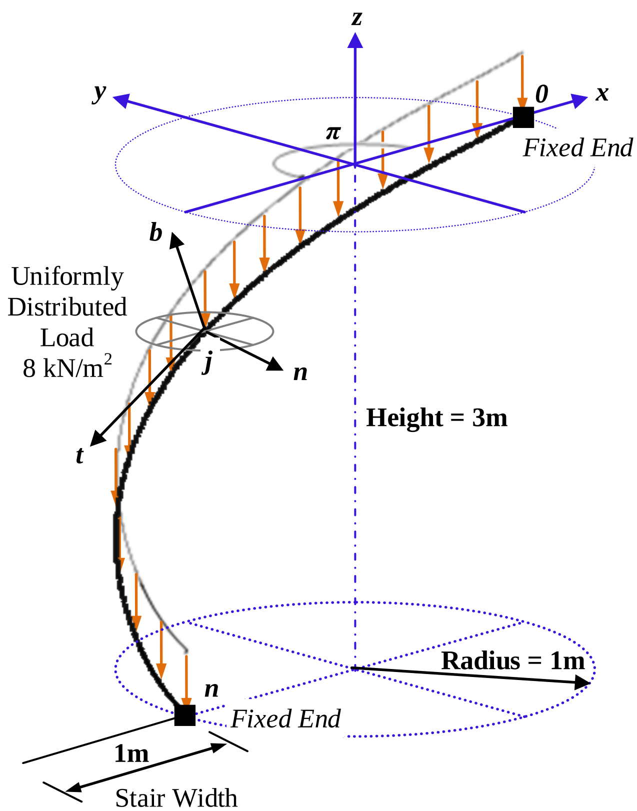

Figure 4: Circular Helical Bi-fixed Beam
Source: Sarria *et al.* (2018).

Parametric equations of the helix axis are as follows,

$x = \cos\theta, y = \sin\theta, z = -3\,\theta/\pi$

At node 0, θ = 0. At node n, θ = π. Interval dθ = π/1000.

Fixed end reactions obtained by applying calculation procedure presented in this article, are compared with the results obtained by Sarria *et al.* [14]. The circular helical beam axis is discretized into 1000 elements (interval dθ = π/1000).

Table 1: Comparision of Fixed End Reactions of Circular Helical Bi-Fixed Beam.

Fixed End Reactions	Present Study	Sarria *et al.* [14]	Percentage Difference
$\hat{f}_{t0}$	−13.847	−13.871	0.17
$\hat{f}_{n0}$	0.656	0.648	−1.23
$\hat{f}_{b0}$	6.792	6.817	0.37
$\hat{m}_{t0}$	−2.693	−2.682	−0.41
$\hat{m}_{n0}$	−4.516	−4.532	0.35
$\hat{m}_{b0}$	1.504	1.516	0.79
$\hat{f}_{tn}$	−17.841	−17.830	−0.06
$\hat{f}_{nn}$	0.656	0.648	−1.23
$\hat{f}_{bn}$	10.975	10.962	−0.12
$\hat{m}_{tn}$	−3.216	−3.224	0.25
$\hat{m}_{nn}$	12.700	12.681	−0.15
$\hat{m}_{bn}$	7.575	7.568	−0.09

Percentage difference in the fixed end reactions as given in table 1 is calculated with reference to the fixed end reactions presented by Sarria *et al.* [14]. To calculate percentage difference, appropriate change of sign (+/−) is taken in to account because axis, initial node and final node are different in present study and in the article of Sarria *et al.* [14].
Comparison of the results (table 1) shows that the fixed end reactions match with the fixed end reactions available in the literature [14]. For this specific problem, results obtained using 1000 elements (interval dθ = π/1000) having the absolute maximum and minimum percentage difference 1.23% and 0.06%, respectively.

5.2 Elliptic-Helical Beam

An elliptic-helical bi-fixed beam with uniform cross-section and material is shown in figure 5. The beam is loaded with uniformly distributed load of −0.1 kip/ft. Following data of the elliptic-helical beam [9] are used to obtain the fixed end reactions:
Semi-major axis, a = 4 ft
Semi-minor axis, b = 3 ft
Constant, c = 1.6 ft
Cross-sectional diameter=1 ft
Elastic modulus, E = 100000 kip/ft^2
Shear modulus, G = 40000 kip/ft^2
Shearing deformation is neglected
Parametric equations of the elliptic-helical beam axis are as follows:

$$x = 4\cos\theta$$
$$y = 3\sin\theta$$
$$z = 1.6\,\theta$$

At node 0, θ = 2π. At node n, θ = 0.
Interval dθ = −π/180

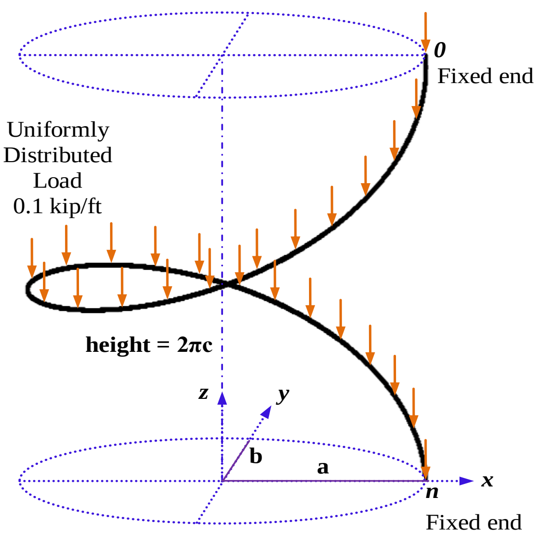

Figure 5: Elliptic-Helical Bi-fixed Beam
Source: Gimena *et al.* (2008).

Fixed end reactions obtained by applying calculation procedure presented in this article, are compared with the results obtained by Gimena *et al.* [9]. The elliptic-helical beam axis is discretized into 360 elements (interval dθ = −π/180).

Table 2: Comparision of Fixed End Reactions of Elliptic-Helical Bi-Fixed Beam.

Fixed End Reactions	**Present Study**	**Gimena *et al.* [9]**	**Percentage Difference**
$\hat{f}_{t0}$	−0.5723	−0.5718	−0.09
$\hat{f}_{n0}$	−0.5612	−0.5612	0.00
$\hat{f}_{b0}$	−1.0668	−1.0721	0.49
$\hat{m}_{t0}$	−1.7789	−1.7991	1.12
$\hat{m}_{n0}$	2.4104	2.4253	0.61
$\hat{m}_{b0}$	0.9622	0.9595	−0.28
$\hat{f}_{tn}$	−0.5698	−0.5718	0.35
$\hat{f}_{nn}$	0.5612	0.5612	0.00
$\hat{f}_{bn}$	−1.0746	−1.0721	−0.23
$\hat{m}_{tn}$	−1.8194	−1.7991	−1.13
$\hat{m}_{nn}$	−2.4399	−2.4253	−0.60
$\hat{m}_{bn}$	0.9570	0.9595	0.26

Percentage difference in the fixed end reactions as given in table 2 is calculated with reference to the fixed end reactions presented by Gimena *et al.* [9]. To calculate percentage difference, appropriate change of sign (+/−) is taken in to account because axis, initial node and final node are different in present study and in the article of Gimena *et al.* [9].
Comparison of the results (table 2) shows that the fixed end reactions match with the fixed end reactions available in the literature [9]. For this specific problem, results obtained using 360 elements (interval dθ = −π/180) having the absolute maximum and minimum percentage difference 1.13% and 0.0%, respectively.
For the statically determinate elliptic-helical beam loaded with uniformly distributed load of −0.1 kip/ft, the graphs of the displacements, rotations, internal forces and internal moment are shown in figure 6, figure 7, figure 8 and figure 9, respectively. The node 0 and node n are freely supported and fixed, respectively.
Comparison of the graphs shown in figure 6 to figure 9 with the graphs available in the article [9] shows that the results are found to match, hence the formulation developed in the present article is verified.
For the elliptic-helical bi-fixed beam loaded with a uniformly distributed load of −0.1 kip/ft, the graphs of the displacements, rotations, internal forces, and internal moment are shown in figure 10, figure 11, figure 12 and figure 13, respectively.

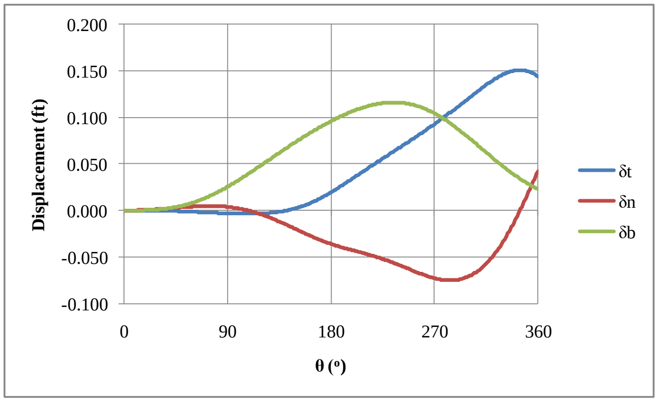

Figure 6: Displacements of Statically Determinate Elliptic-Helical Beam

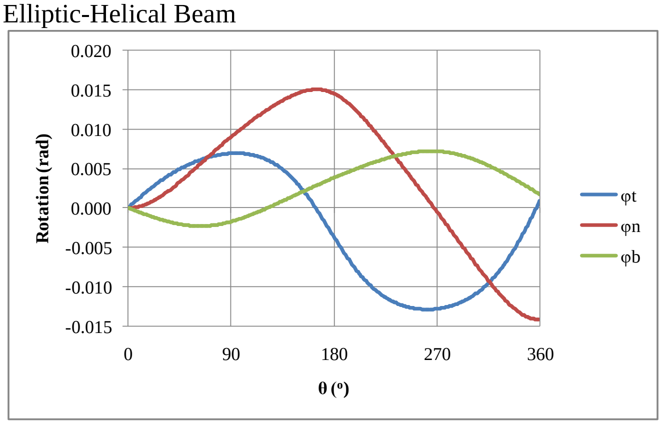

Figure 7: Rotations of Statically Determinate Elliptic-Helical Beam

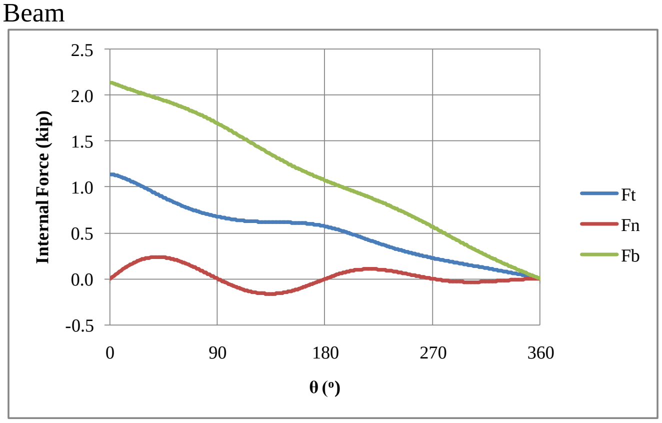

Figure 8: Internal Forces of Statically Determinate Elliptic-Helical Beam

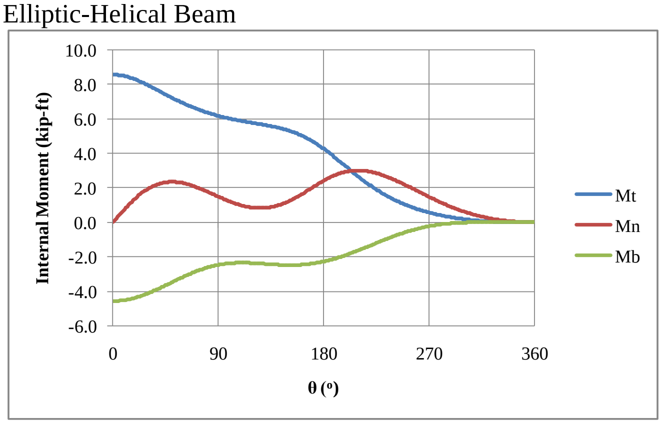

Figure 9: Internal Moments of Statically Determinate Elliptic-Helical Beam

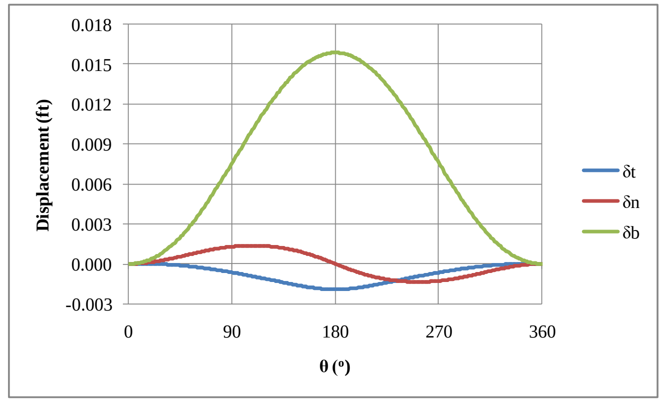

Figure 10: Displacements of Bi-Fixed Elliptic-Helical Beam

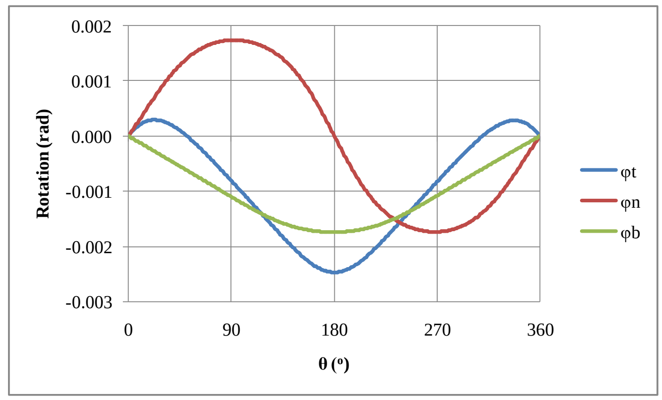

Figure 11: Rotations of Bi-Fixed Elliptic-Helical Beam

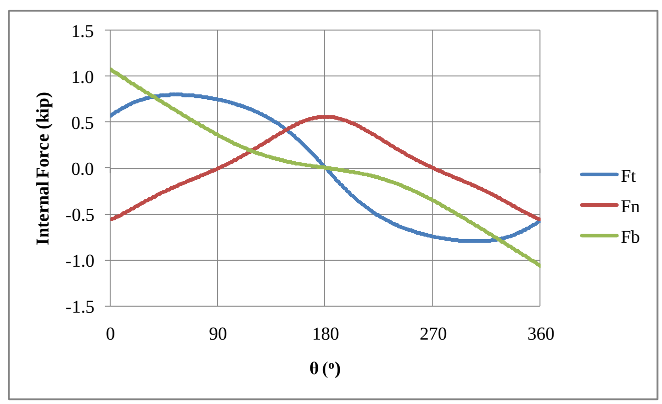

Figure 12: Internal Forces of Bi-Fixed Elliptic-Helical Beam

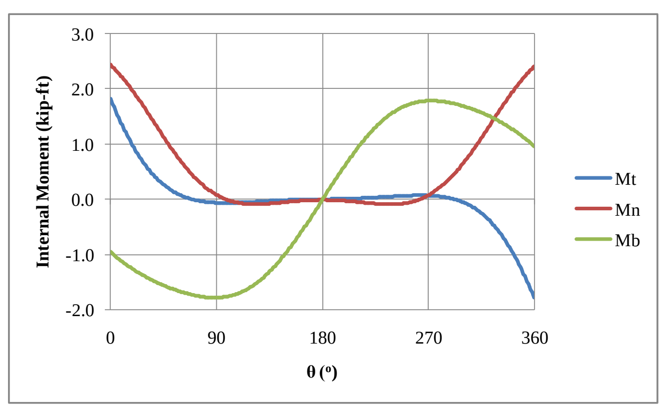

Figure 13: Internal Moments of Bi-Fixed Elliptic-Helical Beam

6. CONCLUSION

Finite difference method along with recursive scheme is easy to implement and gives reasonably accurate results without involving complex calculations. The finite displacement transfer method for the analysis spatial curved bi-fixed beam; presented in this article requires no analytical differentiation and integration of the governing equations. Cartesian coordinates of the curved beam axis, load data, material properties and cross-sectional properties are sufficient to evaluate fixed end reactions, displacements, internal forces and internal moments. No need to develop problem-specific equations of tangent, normal, binormal, flexion curvature and torsion curvature.

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