ISSN 2278-3083
Volume 8, No.6, November –December 2019
International Journal of Science and Applied Information Technology
Available Online at http://www.warse.org/ijsait/static/pdf/file/ijsait16862019.pdf
https://doi.org/10.30534/ijsait/2019/168620198

# NEIGHBORHOODS OF A CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS ASSOCIATED WITH JACKSONS (p; q) DERIVATIVE

**Tariq AL-Hawary[1], A.A. Amourah[2] , Laith Abualigah[3]**
[1]Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan.
[2]epartment of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid, Jordan.
[3]Faculty of Computer Sciences and Informatics, Amman Arab University, Amman, Jordan

## ABSTRACT

By making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the $(n,\delta)$ neighborhoods for a subclass of starlike functions of complex order involving Jacksons $(p,q)$-derivative. Special cases of some of these inclusion relations are shown to yield known results.

**Key words:** Analytic functions, Starlike functions, Convex functions, $(p,q)$ -Derivative, $(n,\delta)$ -Neighborhood, Inclusion relations.

## 1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

(1.1)

which are analytic in the open unit disc $\Delta=\{Z:|Z<1|\}$. Further, let S denote the class of all functions □□2 A which are univalent in $\Delta$ (for details, see [8]; see also some of the recent investigations [2, 4, 5, 6, 10, 18]).

Denote by T a subclass of A consisting functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n, \qquad a_n\geq 0, \ Z\in\Delta$$

(1.2)

introduced and studied by Silverman [17].

We briefly recall here the notion of q-operators i.e. q-difference operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of q-calculus was first introduced by Jackson [11,21,22]. Kanas and Raducanu [14] have used the fractional q-calculus operators in investigations of certain classes of functions which are analytic in $\Delta$. For details on q-calculus one can refer [3, 7, 11, 13, 14, 19, 20] and also the reference cited therein. For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0<p<q\leq 1$.

For $0<p<q\leq 1$ the Jacksons (p□□q)-derivative of a function □□2 A is, by definition, given as follows [11]

$$D_{p,q}f(z)=\begin{cases}\dfrac{f(pz)-f(qz)}{(p-q)z} & \text{for } z\neq 0,\\ f'(0) & \text{for } z=0.\end{cases}$$

(1.3)

From (1.3), we have

$$D_q f(z)=1+\sum_{n=2}^{\infty}[n]_{p,q}\,a_n z^{n-1}$$

(1.4)

where

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},$$

(1.5)

is called (p, q)-bracket or twin-basic number. Clearly for a function □(□)=z^n□□we obtain

$$D_{p,q}h(z)=D_{p,q}z^n=\frac{p^n-q^n}{p-q}z^{n-1}=[n]_{p,q}\,z^{n-1}.$$

Note also that for =1, the Jackson (p, q)-derivative reduces to the Jackson q-derivative given by (see [11]).

we define the Salagean (p; q)-differential operator as follows:

$$D^{0}{}_{p,q}f(z)=f(z)$$
$$D^{1}{}_{p,q}f(z)=zD_{p,q}f(z)$$
$$\vdots$$
$$D^{m}{}_{p,q}f(z)=zD^{1}{}_{p,q}\left(D^{m-1}{}_{p,q}f(z)\right)$$
$$=z+\sum_{n=2}^{\infty}[n]^{m}_{p,q}a_nz^n \quad (m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, z\in\Delta)$$

(1.6)

We note that if p = 1 and $\lim_q$□! 1¯□□we obtain the familiar Salagean derivative [16]

$$D^mf(z)=\sum_{n=2}^{\infty}n^ma_nz^n \quad (m\in\mathbb{N}_0; z\in\Delta).$$

(1.7)

Now let

$$\Re^{0,m}_{\lambda,p,q}f(z)=D^m_{p,q}f(z),$$
$$\Re^{1,m}_{\lambda,p,q}f(z)=(1-\lambda)D^m_{p,q}f(z)+\lambda z(D^m_{p,q}f(z))'$$
$$=z+\sum_{n=2}^{\infty}[n]^m_{p,q}[1+(n-1)\lambda]a_nz^n,$$
$$\Re^{2,m}_{\lambda,p,q}f(z)=(1-\lambda)\Re^{1,m}_{\lambda,p,q}f(z)+\lambda z(\Re^{1,m}_{\lambda,p,q}f(z))'$$
$$=z+\sum_{n=2}^{\infty}[n]^m_{p,q}[1+(n-1)\lambda]^2a_nz^n.$$

(1.8)

In general, we have

$$\Re^{\zeta,m}_{\lambda,p,q}f(z)=(1-\lambda)\Re^{\zeta-1,m}_{\lambda,p,q}f(z)+\lambda z(\Re^{\zeta-1,m}_{\lambda,p,q}f(z))'$$
$$=z+\sum_{n=2}^{\infty}[n]^m_{p,q}[1+(n-1)\lambda]^{\zeta}a_nz^n \quad (\lambda>0;\zeta,m\in\mathbb{N}_0).$$

Clearly, we have $\Re^{0,0}_{\lambda,p,q}f(z)=f(z)$ and $\Re^{1,0}_{1,p,q}f(z)=zf'(z)$.

We note that when p = 1; we get the differential operator $\Re^{\zeta,m}_{\lambda,q}f(z)$ defined and studied by Frasin and Murugusundaramoorthy [9]. Also, We note that when p = 1 and lim□!1¯□□we get the differential operator

$$\Re^{\zeta,m}_{\lambda}f(z)=z+\sum_{n=2}^{\infty}n^m[1+(n-1)\lambda]^{\zeta}a_nz^n \quad (\lambda>0;\zeta,m\in\mathbb{N}_0).$$

With the aid of the differential operator $\Re^{\zeta,m}_{\lambda,p,q}f(z)$ □□we say that a function □(□) belonging to A is said to be in the class $S^{\zeta,m}_{\lambda,p,q}(b,\alpha)$ if it satisfies

$$\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{z\left(\Re^{\zeta,m}_{\lambda,p,q}f(z)\right)'}{\Re^{\zeta,m}_{\lambda,p,q}f(z)}-\alpha\right)\right\}>\left|1+\frac{1}{b}\left(\frac{z\left(\Re^{\zeta,m}_{\lambda,p,q}f(z)\right)'}{\Re^{\zeta,m}_{\lambda,p,q}f(z)}-1\right)\right|, \quad (z\in\Delta)$$

(1.9)

Where

$0<\alpha\le 1,\beta\ge 0,\lambda>0,\zeta,m\in\mathbb{N}_0$ and $b\in\mathbb{C}^*=\mathbb{C}-\{0\}$.

Further, we de.ne the class $ST^{\zeta,m}_{\lambda,p,q}(b,\alpha)$ by

$$ST^{\zeta,m}_{\lambda,p,q}(b,\alpha)=S^{\zeta,m}_{\lambda,p,q}(b,\alpha)\cap T.$$

(1.10)

## 2. COEFFICIENT INEQUALITIES

A necessary and sufficient condition for a function to be in the class $ST^{\zeta,m}_{\lambda,p,q}(b,\alpha)$ is given by:

**Lemma 2.1.** [1] Let the function f(z) be defined by (1.2): Then □(□)2 $ST^{\zeta,m}_{\lambda,p,q}(b,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}\left[(n+|b|)(1-\beta)+\beta-\alpha\right][n]^m_{p,q}[1+(n-1)\lambda]^{\zeta}|a_n||z^n|\le 1-\alpha+|b|(1-\beta),$$

(2.1)

where $-1\le\alpha<1,\beta\ge 0$ and $b\in\mathbb{C}^*$.

**Corollary 2.2.** [1] Let the function □(□)2 $ST^{\zeta,m}_{\lambda,p,q}(b,\alpha)$. Then

$$|a_n|\le\frac{1-\alpha+|b|(1-\beta)}{\left[(n+|b|)(1-\beta)+\beta-\alpha\right][n]^m_{p,q}[1+(n-1)\lambda]^{\zeta}},$$

$n\ge 2,-1\le\alpha<1,\beta\ge 0$ and $b\in\mathbb{C}^*$, with equality for

$$f(z)=z-\frac{1-\alpha+|b|(1-\beta)}{\left[(n+|b|)(1-\beta)+\beta-\alpha\right][n]^m_{p,q}[1+(n-1)\lambda]^{\zeta}}z^n.$$

## 3. NEIGHBORHOOD

The concept of $(n,\delta)$-neighborhood was first introduced by Goodman [12], and then generalized by Ruscheweyh [15]. The $(n,\delta)$-neighborhood of the function □2 T is defined by

$$N_{n,\delta}(f)=\left\{g\in T: g(z)=z-\sum_{n=2}^{\infty}|b_n|z^n \text{ and } \sum_{n=2}^{\infty}n|a_n-b_n|\le\delta\right\}$$

. (3.1)

In particular, for the identity function e(z)=z, we have

$$N_{n,\delta}(e)=\left\{g\in T: g(z)=z-\sum_{n=2}^{\infty}|b_n|z^n \text{ and } \sum_{n=2}^{\infty}n|b_n|\le\delta\right\}. \quad (3.2)$$

**Theorem 3.1.** If

$$\delta=\frac{2\left[1-\alpha+|b|(1-\beta)\right]}{\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}}, \quad (3.3)$$

then

$$ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)\subset N_{n,\delta}(e). \quad (3.4)$$

***Proof***. Let □(□)2 $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$. Lemma 2.1 yields

$$\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}\sum_{n=2}^{\infty}|a_n|\le 1-\alpha+|b|(1-\beta),$$

which yields

$$\sum_{n=2}^{\infty}|a_n|\frac{1-\alpha+|b|(1-\beta)}{\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}}. \quad (3.5)$$

On the other hand, use of (2.1), in conjunction with (3.5), we have

$$(1-\beta)[2]_{p,q}^{m}[1+\lambda]^{\zeta}\sum_{n=2}^{\infty}n|a_n|$$

$$\le 1-\alpha+|b|(1-\beta)+\left[(\alpha-\beta)-|b|(1-\beta)\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}\sum_{n=2}^{\infty}|a_n|$$

$$\le\frac{2(1-\beta)+\left[(1-\alpha)+|b|(1-\beta)\right]}{(2+|b|)(1-\beta)+\beta-\alpha}.$$

Hence

$$\sum_{n=2}^{\infty}n|a_n|\le\frac{2\left[1-\alpha+|b|(1-\beta)\right]}{\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}}=\delta,$$

which, by the definition (3.2), establishes the inclusion (3.4) asserted by Theorem 3.1.

Now we determine the neighborhood for the class $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$ □□which we define as follows. A function □(□)2 T is said to be in the class $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$ if there exists a function □(□)2 $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\varpi \qquad (z\in\Delta; 0\le\varpi<1). \quad (3.6)$$

**Theorem 3.2.** If □□2 $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$ □and

$$\varpi=1-\frac{\delta\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}}{2\left[((2+|b|)(1-\beta)+\beta-\alpha)[2]_{p,q}^{m}[1+\lambda]^{\zeta}-((1-\alpha)+|b|(1-\beta)\right]}, \quad (3.7)$$

then

$$N_{n,\delta}(g)\subset ST_{\lambda,p,q}^{\zeta,m}(b,\alpha).$$

***Proof***. Suppose that □□2 $N_{n,\delta}(g)$. We find from (3.1) that

$$\sum_{n=2}^{\infty}n|a_n-b_n|\le\delta, \quad (3.8)$$

which implies that

$$\sum_{n=2}^{\infty}|a_n-b_n|\le\frac{\delta}{2}.$$

Next, since □□2 $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$, we have [cf. equation 3.5]

$$\sum_{n=2}^{\infty}|b_n|\le\frac{2\left[1-\alpha+|b|(1-\beta)\right]}{\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}}.$$

Letting j□j ! 1□□so

$$\left|\frac{f(z)}{g(z)}-1\right|\le\frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}|b_n|}$$

$$\le\frac{\delta}{2}\left(\frac{\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}}{\left[(2+|b|)(1-\beta)+\beta-\alpha\right][2]_{p,q}^{m}[1+\lambda]^{\zeta}-\left[1-\alpha+|b|(1-\beta)\right]}\right)\le 1-\varpi$$

provided that □□is given by (3.7). Thus, by the above definition, $ST_{\lambda,p,q}^{\zeta,m}(b,\alpha)$

## REFERENCES


1. Al-Hawary, T., Yousef, F., Frasin, B.A. Subclasses of analytic functions of complex order involving Jackson.s (p; q)-derivative, Proceedings of International Conference on Fractional Differentiation and its Applications (ICFDA). Available at SSRN 3289803 (2018).
2. Al-Hawary, T., Frasin, B.A., Yousef, F. Coefficients estimates for certain classes of analytic functions of complex order, Afrika Matematika 29(7-8) (2018): 1265-1271.

3. Al-Oboudi, F.M. On univalent functions de.ned by a generalized S¼al¼agean operator, International Journal of Mathematics and Mathematical Sciences 2004(27) (2004): 1429-1436.
4. Amourah, A.A., Yousef, F., Al-Hawary, T., Darus, M. On a class of p-valent non-Bazilevic functions of order $\mu+i\beta$ ,International Journal of Mathematical Analysis 10(15) (2016): 701-710. https://doi.org/10.12988/ijma.2016.6236
5. Amourah, A.A., Yousef, F., Al-Hawary, T., Darus, M. On H3(p) Hankel determinant for certain subclass of p-valent functions, Italian Journal of Pure and Applied Mathematics 37 (2017): 611-618.
6. Tariq Al-Hawary. 2018. A Certain New Familiar Class of Univalent Analytic Functions with Varying Argument of Coefficients Involving Convolution. Italian journal of pure and applied mathematics-N. 39(2018):326–333.
7. Aral, A., Gupta, V., Agarwal, R.P. Applications of q-calculus in operator theory, Springer, New York, 2013.
8. Duren, P.L. Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
9. Frasin, B.A., Murugusundaramoorthy, G. A subordination results for a class of analytic functions defined by q-di¤erential operator, submitted.
10. Frasin, B.A., Al-Hawary, T., Yousef, F. Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, Afrika Matematika (2018) 1-8.
11. Jackson, F.H. On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh 46 (1908): 253-281.
12. Goodman, A.W. Univalent functions and nonanalytic curves, Proceedings of the American Mathematical Society 8(3) (1957): 598-601.
13. Govindaraj, M., Sivasubramanian, S. On a class of analytic function related to conic domains involving q-calculus, Analysis Math. 43(3) (2017): 475-487. https://doi.org/10.1007/s10476-017-0206-5
14. Kanas, S., Raducanu, D. Some subclass of analytic functions related to conic domains, Math. Slovaca 64(5) (2014): 1183-1196.
15. Ruscheweyh, S. Neighborhoods of univalent functions, Proceedings of the American Mathematical Society 81(4) (1981): 521-527.
16. G. Salagean, Subclasses of univalent functions, in . Complex Analysis: Fifth Romanian Finnish Seminar,.Part I(Bucharest, 1981), pp.362-372, Lecture Notes in Mathematics, Vol. 1013, Springer- Verlag, Berlin/ New York, 1983.
17. Silverman, H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975): 109-116.
18. Yousef, F., Amourah, A.A. Darus, M. Differential sandwich theorems for p-valent functions associated with a certain generalized differential operator and integral operator, Italian Journal of Pure and Applied Mathematics 36 (2016): 543-556.
19. Yousef, F., Al-Hawary, T., Murugusundaramoorthy, G. 2019. Fekete–Szegö functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator. Afrika Matematika, 30(3-4), 495-503.
20. Amourah, A.A., Yousef, F., Al-Hawary, T., Darus, M. 2017. On H3(p) Hankel determinant for certain subclass of p-valent functions. Italian journal of pure and applied mathematics -N. 37(2017): 611-618.
21. Abualigah, L. M. Q. (2019). Feature selection and enhanced krill herd algorithm for text document clustering. Berlin: Springer.
22. Abualigah, L. M., Khader, A. T., & Hanandeh, E. S. (2019). Modified Krill Herd Algorithm for Global Numerical Optimization Problems. In *Advances in Nature-Inspired Computing and Applications* (pp. 205-221). Springer, Cham. https://doi.org/10.1007/978-3-319-96451-5_9